EHP sequences, exponents of homotopy groups and the Barratt conjecture

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Jie Wu

Hebei Normal University

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EHP sequences

Exponents of Homotopy Groups

The Barratt Conjecture

James' Construction

James' construction: Hatcher's book, section 3.2, p.224.

$J(X)$=free monoid generated by X with single relation that the basepoint $* = 1$, having weak topology.

Suspension Splitting Theorem of the James Construction: Hatcher's book, Section 4I, Proposition 4I.2, p.468.

$$\Sigma J(X) \simeq \bigvee_{n=1}^{\infty} \Sigma X^{\wedge n}.$$

$J(X) \simeq \Omega\Sigma X$ for path-connected CW-complex. Hatcher's book, Section 4J, Theorem 4J.1, p.471. In Hatcher's book, Section 4J, he mentions a little about EHP sequence.

James-Hopf Maps and EHP fibre sequence

Hilton-Hopf Invariants: Whitehead's book, Chapter XI, Section 8, p.533-541. Also Neisendorfer's book (Algebraic Methods in Unstable Homotopy Theory), Chapters 4, 5.

James-Hopf map $H_k\colon J(X) \to J(X^{\wedge k})$:
$x_1 x_2 \cdots x_n \mapsto \prod_{1\leq i_1<i_2<\cdots<i_k\leq n} x_{i_1} \wedge \cdots \wedge x_{i_k}$, right lexicographic order.

Homework (EHP Fibre Sequence Theorem). There is a fibre sequence $S^{n-1} \xrightarrow{E} \Omega S^n \xrightarrow{H=H_2} \Omega S^{2n-1}$ localized at 2.

Hint: Let F be the homotopy fibre of $H_2\colon \Omega S^n \to \Omega S^{2n-1}$. Show that $F \simeq S^{n-1}$: Step 1. Show that, in mod 2 homology, $H^*(\Omega S^n) \cong H^*(\Omega S^{2n-1}) \otimes H^*(S^{n-1})$ is a free $H^*(\Omega S^{2n-1})$-module. Step 2. Work on the Serre cohomology spectral sequence.

Hopf invariant one problem: When is S^n an H-space?

EHP Sequences for Odd Primes

Reference: Neisendorfer's book (Algebraic Methods in Unstable Homotopy Theory), Chapters 5.

There is a fibre sequence
$J_{p-1}(S^{2n}) \xrightarrow{E} J(S^{2n}) \simeq \Omega S^{2n+1} \xrightarrow{H_p} J(S^{2np}) \simeq \Omega S^{2np+1}$.
(Neisendorfer's book Corollary 5.3.4, p.148.)

Toda Fibre Sequence. $S^{2n-1} \longrightarrow \Omega J_{p-1}(S^{2n}) \xrightarrow{H} \Omega S^{2np-1}$.
(Neisendorfer's book Section 5.4.)

The Functor $A^{\min}$

References:

- Selick, Paul; Wu, Jie On natural coalgebra decompositions of tensor algebras and loop suspensions. Mem. Amer. Math. Soc. 148 (2000), no. 701, viii+109 pp.
- Selick, Paul; Wu, Jie The functor $A^{\min}$ on p-local spaces. Math. Z. 253 (2006), no. 3, 435-451.
- Selick, Paul; Theriault, Stephen; Wu, Jie Functorial homotopy decompositions of looped co-H spaces. Math. Z. 267 (2011), no. 1-2, 139-153.
- Grbić, J.; Theriault, S.; Wu, J. Decompositions of looped co-H-spaces. Proc. Amer. Math. Soc. 141(2013), no. 4, 1451-1464.

For path-connected p-local co-H-space X, $\bar{A}^{\min}(X)$ is the **functorial smallest retract** of ΩX containing the bottom cell.

EHP Sequences for Finite Complexes

Reference:

- Wu, J. The functor $A^{\min}$ for $(p-1)$-cell complexes and EHP sequences. Israel J. Math. 178 (2010),349-391.

$H_*(\ )$ means mod p homology. $b_Y = \sum_{q=0}^{\infty} q\dim\tilde{H}_q(Y;\mathbb{Z}/p)$.

EHP **sequence.** Let Y be p-local simply connected such that $\tilde{H}_{\mathrm{odd}}(Y)=0$ and $\dim\tilde{H}_*(Y)=p-1$. There is a fibre sequence $\bar{E}(Y)\xrightarrow{E}\bar{A}^{\min}(Y)\xrightarrow{H_p}\bar{A}^{\min}(\Sigma^{b_Y-p+1}Y)$, where $H_*(\bar{E}(Y))=E(s^{-1}\tilde{H}_*(Y))$.

Moreover the connecting map $P\colon\Omega\bar{A}^{\min}(\Sigma^{b_Y-p+1}Y)\to\bar{E}(Y)$ factors through the bottom cell of $\bar{E}(Y)$.

Generalized Hopf Invariant One Problems

Reference:

- Grbić, Jelena; Harper, John; Mimura, Mamoru; Theriault, Stephen; Wu, Jie Rank $p-1$ mod-p H-spaces. Israel J. Math. 194 (2013), no. 2, 641-688.

Question 1. Let X be a simply connected CW-complex such that $H_*(X;\mathbb{Z}/p)$ is the exterior algebra of rank $p-1$ with generators in odd dimensions. Find a criterion when X is an H-space localized at p.

Question 1—special case. For $p=3$, consider X given as the total space of spherical fibration over a sphere. Examples, $X = Sp(n+1)/Sp(n-1)$.

Homotopy Groups and Spherical Fibrations over Spheres

Let $f\colon S^{2m+2k} \to S^{2m+1}$ represent an element $[f] \in \pi_{2m+2k}(S^{2m+1})$ localized at 3. Let $E([f]) = \bar{E}(\Sigma C_f)$ and $A([f]) = \bar{A}^{\min}(\Sigma C_f)$.

There is a fibre sequence $S^{2m+1} \to E([f]) \to S^{2m+2k+1}$ with the connecting map $P_f\colon \Omega S^{2m+2k+1} \to S^{2m-1}$ an extension of f.

$\pi_*(E([f])$ are built by cokernels and kernels of $P_{f*}\colon \pi_*(\Omega S^{2m+2k+1}) \to \pi_*(S^{2m+1})$.

There is a fibre sequence $E([f]) \to A([f]) \to A(\Sigma^{4m+2k+2}[f])$.

Question 2 (Homotopy Group Problem). Compute $\pi_*(E([f]))$, $\pi_*(A([f])$ and their connections with homotopy groups of spheres.

Homotopy Groups and Spherical Fibrations over Spheres

Question 3 (Hopf Invariant Problem). Does there exist a homotopy commutative diagram

$$\begin{array}{ccc} \Omega S^{2m+1} & \xrightarrow{H_p} & \Omega S^{2pm+1} \\ \downarrow & & \downarrow \\ \Omega E([f]) & \dashrightarrow & \Omega E([\theta_f]) \\ \downarrow & & \downarrow \\ \Omega S^{2m+2k+1} & \xrightarrow{H_p} & \Omega S^{2p(m+k)+1} \end{array}$$

for some $[\theta_f] \in \pi_{2p(m+k)}(S^{2pm+1})$.

Homotopy Groups and Spherical Fibrations over Spheres

Question 4 (Geometric Analysis on Homotopy). Does there exist a manifold $M([f])$ such that $M([f])_{(3)} \simeq E([f])$? If so, classifying the elements $[f]$ using geometric properties.

Note. $E([f])$ is only defined as a 3-local space.

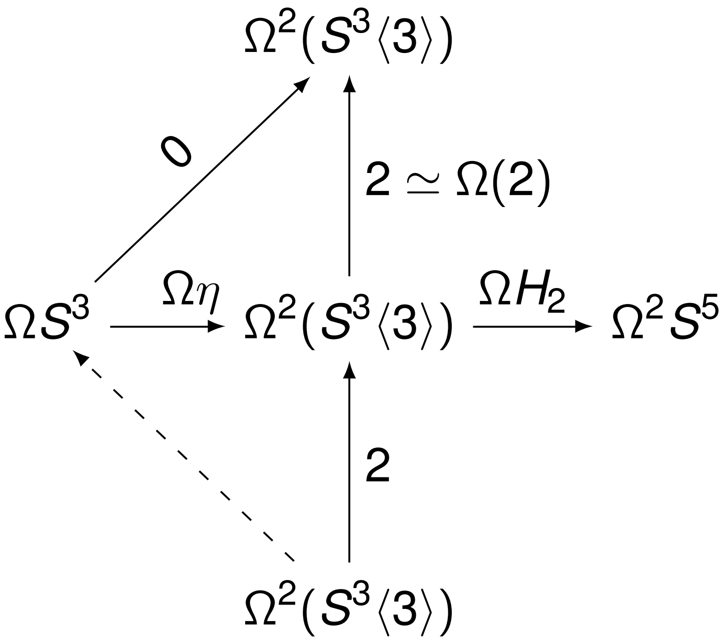
The canonical suspension $\Omega\Sigma X \to \Omega^2\Sigma^2 X$ induces $\bar{A}^{\mathrm{min}}(\Sigma X) \to \Omega\bar{A}^{\mathrm{min}}(\Sigma^2 X)$. It then should induce a double-suspension $E[f] \to \Omega^2 E(\Sigma^2[f])$.

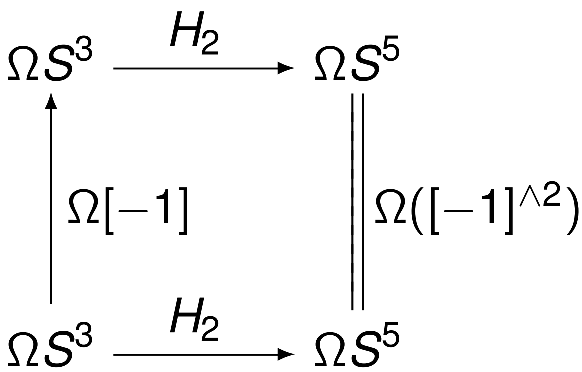
Question 5 (Double Suspension Problem). Let $W([f])$ be the homotopy fibre of $E([f]) \to \Omega^2 E(\Sigma^2[f])$. Determine the exponents of $\pi_*(W([f]))$.

Question 6. Explore homotopy theory of $\bar{A}^{\mathrm{min}}$ on p-cell co-H-spaces.

Example: $4 \cdot \mathrm{Tor}_2(\pi_*(S^3)) = 0$

$\pi_4(S^3) = \mathbb{Z}/2$ using blackboard.





Classical Theorems

Selick, Paul A decomposition of $\pi_*(S^{2p+1}, \mathbb{Z}/p\mathbb{Z})$, Topology 20 (1981), no. 2, 175-177: $p \cdot \mathrm{Tor}_p(\pi_*(S^3)) = 0$ for $p > 2$.

$$\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3\langle 3\rangle_{(p)} \times C(p)_{(p)}$$

Cohen, F. R.; Moore, J. C.; Neisendorfer, J. A. Torsion in homotopy groups. Ann. of Math. (2) 109 (1979), no. 1, 121-168.
Cohen, F. R.; Moore, J. C.; Neisendorfer, J. A. The double suspension and exponents of the homotopy groups of spheres. Ann. of Math. (2) 110 (1979), no. 3, 549-565.

$$p^n \cdot \mathrm{Tor}_p(\pi_*(S^{2n+1}) = 0$$

for $p > 3$. **Method.** $p\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$ factors through S^{2n-1}.

Barratt Conjecture

Barratt Conjecture. Suppose that $[p^r] \simeq *\colon \Sigma^2 X \to \Sigma^2 X$. Then $p^{r+1} \simeq *\colon \Omega^2\Sigma^2 X \to \Omega^2\Sigma^2 X$. In particular, $p^{r+1} \cdot \pi_*(\Sigma^2 X) = 0$.

Barratt Conjecture on Mapping. Let $f\colon \Sigma^2 X \to Y$ be a map. Suppose that $[f]$ is of order bounded by p^r in $[\Sigma^2 X, Y]$. Then $[f]$ is of order p^{r+1} in the group $[\Omega^2\Sigma^2 X, \Omega^2 Y]$.

Cohen Program

References:

- Cohen, F. R. On combinatorial group theory in homotopy. Homotopy theory and its applications (Cocoyoc, 1993), 57-63, Contemp. Math., 188, Amer. Math. Soc., Providence, RI, 1995.
- Selick, Paul; Wu, Jie On natural coalgebra decompositions of tensor algebras and loop suspensions. Mem. Amer. Math. Soc. 148 (2000), no. 701, viii+109 pp.
- Wu, Jie Homotopy theory of the suspensions of the projective plane. Mem. Amer. Math. Soc. 162 (2003), no. 769, x+130 pp.
- Wu, Jie On maps from loop suspensions to loop spaces and the shuffle relations on the Cohen groups. Mem. Amer. Math. Soc. 180 (2006), no. 851, vi+64 pp.

Now use blackboard for details.

Thank You!